NON-ONE-DIMENSIONAL SOLUTIONS OF THE EQUATIONS OF MAGNETIC GAS DYNAMICS

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To supplement previous reports on one-dimensional axisymmetric motions of an electrically conducting gas in a magnetic field, when the velocity can be represented by a formula of type $v = r\dot\Phi(t)$, the paper considers flows for which the velocity components depend linearly on the distance coordinates, but where the flow is no longer one-dimensional. The motion of the gas is assumed to be two-dimensional, and the magnetic field is taken as perpendicular to the plane of flow. The solution obtained depends on an arbitrary function and several arbitrary constants. The case is considered where a uniform expansion of the gas takes place along the magnetic field, at a two-dimensional flow transverse to it.

1. The system of equations of magnetic gas dynamics (Bibl.1, 2) describing two-dimensional nonstationary motion of the gas in a magnetic field perpendicular to the plane of flow, has the form[*]   /946

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(p + \frac{H^2}{8\pi}\right),$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial}{\partial y}\left(p + \frac{H^2}{8\pi}\right),$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \ldots + t_0\,\rho = 0,$$ (1.1)

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0,$$

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[*] Everywhere in this Section, we assume $\nu = 0$.

[**] Numbers in the margin indicate pagination in the original foreign text.

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \gamma p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \gamma p \frac{v}{t + t_0} = 0.$$

where u, v are the components of the velocity vector, p the pressure, ρ the density, H the magnetic field strength, t the time, and x, y rectangular Cartesian coordinates. The gas is assumed to possess infinite electric conductivity.

Let us introduce new sought functions and new independent variables by means of the relations

$$\tau = t, \qquad \xi = \frac{x}{f(t)}, \qquad \eta = \frac{y}{\psi(t)},$$
$$u_1 = u - \frac{f'}{f} x, \qquad v_1 = v - \frac{\psi'}{\psi} y. \qquad (1.2)$$

The quantities p, ρ and H will be left as before. For the operations of differentiation we will have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{f'}{f^2} x \frac{\partial}{\partial \xi} - \frac{\psi'}{\psi^2} y \frac{\partial}{\partial \eta},$$
$$\frac{\partial}{\partial x} = \frac{1}{f} \frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial y} = \frac{1}{\psi} \frac{\partial}{\partial \eta}. \qquad (1.3)$$

If the system of equations (1.1) is transformed to a new form by means of eqs.(1.2), (1.3) and if we put $u_1 = v_1 = 0$, we will obtain

$$f'' \xi = -\frac{1}{f \rho} \frac{\partial}{\partial \xi} \left(p + \frac{H^2}{8\pi} \right), \qquad \psi'' \eta = -\frac{1}{\psi \rho} \frac{\partial}{\partial \eta} \left(p + \frac{H^2}{8\pi} \right),$$
$$\frac{\partial \rho}{\partial \tau} + \rho \left(\frac{f'}{f} + \frac{\psi'}{\psi} \right) = 0, \quad \frac{\partial p}{\partial \tau} + \gamma p \left(\frac{f'}{f} + \frac{\psi'}{\psi} \right) = 0, \quad \frac{\partial H}{\partial \tau} + H \left(\frac{f'}{f} + \frac{\psi'}{\psi} \right) = 0. \qquad (1.4)$$

The last three equations of the system (1.4) yield

$$\rho = \frac{\rho_0(\xi, \eta)}{f \psi}, \qquad H = \frac{h_0(\xi, \eta)}{f \psi}, \qquad p = \frac{p_0(\xi, \eta)}{f^\gamma \psi^\gamma}.$$

where $\rho_0(\xi, \eta)$, $h_0(\xi, \eta)$ and $p_0(\xi, \eta)$ are arbitrary functions. Substituting the expressions for ρ, H, and p into the first two equations of the system (1.4), we obtain

$$f'' \xi = -f^{-\gamma} \psi^{1-\gamma} \frac{1}{\rho_0} \frac{\partial p_0}{\partial \xi} - f^{-2} \psi^{-1} \frac{1}{\rho_0} \frac{\partial}{\partial \xi} \left(\frac{h_0^2}{8\pi} \right). \qquad (1.5)$$

$$\psi^{\cdot}\eta = -\psi^{-\gamma}f^{1-\gamma}\frac{1}{\rho_0}\frac{\partial p_0}{\partial\eta} - \psi^{-2}f^{-1}\frac{1}{\rho_0}\frac{\partial}{\partial\eta}\left(\frac{h_0^2}{8\pi}\right).$$

The variables are separated if we assume that

$$\frac{1}{\rho_0}\frac{\partial p_0}{\partial\xi} = an\xi, \qquad \frac{1}{\rho_0}\frac{\partial p_0}{\partial\eta} = bn\eta,$$
$$\frac{1}{\rho_0}\frac{\partial}{\partial\xi}\left(\frac{h_0^2}{8\pi}\right) = am\xi, \qquad \frac{1}{\rho_0}\frac{\partial}{\partial\eta}\left(\frac{h_0^2}{8\pi}\right) = bm\eta. \tag{1.6}$$

To determine the functions f and ψ we have the system of equations

$$f^{\cdot} = -af^{-2}\psi^{-1}(nf^{2-\gamma}\psi^{2-\gamma} + m), \quad \psi^{\cdot} = -b\psi^{-2}f^{-1}(nf^{2-\gamma}\psi^{2-\gamma} + m) \tag{1.7}$$

where a, b, n and m are arbitrary constants.

The system of equations (1.6) is easily integrated, yielding

$$p_0 = d + nF\left(\frac{a}{2}\xi^2 + \frac{b}{2}\eta^2\right), \qquad \rho_0 = F'\left(\frac{a}{2}\xi^2 + \frac{b}{2}\eta^2\right),$$
$$h_0^2 = 8\pi\left[c + mF\left(\frac{a}{2}\xi^2 + \frac{b}{2}\eta^2\right)\right]$$

where F is an arbitrary function, and d and c are arbitrary constants.

Thus, we have

$$p = f^{-\gamma}\psi^{-\gamma}\left[d + nF\left(\frac{a}{2}\xi^2 + \frac{b}{2}\eta^2\right)\right], \quad \rho = f^{-1}\psi^{-1}F'\left(\frac{a}{2}\xi^2 + \frac{b}{2}\eta^2\right),$$
$$H^2 = 8\pi f^{-2}\psi^{-2}\left[c + mF\left(\frac{a}{2}\xi^2 + \frac{b}{2}\eta^2\right)\right], \quad u_1 = v_1 = 0.$$

Returning to the original variables and using eqs.(1.2), we obtain the following solution of the system of equations (1.1):

$$u = \frac{f'}{f}x, \quad v = \frac{\psi'}{\psi}y, \quad p = f^{-\gamma}\psi^{-\gamma}[d + nF(\alpha)],$$
$$H^2 = 8\pi f^{-2}\psi^{-2}[c + mF(\alpha)], \quad \rho = f^{-1}\psi^{-1}F'(\alpha), \quad \alpha = \frac{a}{2}\frac{x^2}{f^2} + \frac{b}{2}\frac{y^2}{\psi^2}. \tag{1.8}$$

Since the functions f and ψ enter into the system of equations (1.7), which is to some degree dependent on γ, only the positive solutions of the system of equations (1.7) need be considered. From the condition that ρ be positive, it follows that the function $F(\alpha)$ must be such that the inequality $F'(\alpha) > 0$ is satisfied. It follows from the expressions for pressure and density that

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$$\frac{p}{\rho^\gamma} = \Phi(\alpha) = \frac{d + nF(\alpha)}{[F'(\alpha)]^\gamma}.$$

Hence, it is clear that the arbitrary function $F(\alpha)$ is determined by prescribing the entropy distribution among the fluid particles at a certain instant of time.

A cylindrical surface on which, at a certain instant of time t_0, the pressure, density, and magnetic field strength are all equal is of the form

$$\frac{a}{2}\frac{x^2}{f^2} + \frac{b}{2}\frac{y^2}{\psi^2} = \alpha_1 \qquad (\alpha_1 = \text{const}).$$

On any of such cylindrical surfaces the total pressure, which is the sum of the magnetic and hydrodynamic pressures, will depend only on the time t:

$$p_* = p + \frac{H^2}{8\pi} = f^{-\gamma}\psi^{-\gamma}[d + nF(\alpha_1)] + \psi^{-\gamma}f^{-\gamma}[c + mF(\alpha_1)].$$

It is easy to demonstrate that the surface under consideration will at all times consist of the very same fluid particles. On the basis of the above, such a surface may be taken as the free boundary of the conducting gas in the magnetic field. In this case the total pressure p_* on the boundary will be balanced by the pressure of the external magnetic field $H_1^2/8\pi$, i.e.,

$$p_* = H_1^2/8\pi. \tag{1.9}$$

From this we find the intensity of the external magnetic field as a function of time.

Depending on the values of the parameters a, b, c, and α, this boundary may be an ellipse, a hyperbola, or a pair of intersecting straight lines.

Let us consider the case when $a > 0$, $b > 0$, $n > 0$ and $m > 0$.

Let the function $F(\alpha)$ be a continuous monotonically increasing function, bounded at zero. From the condition that the hydrodynamic pressure p and the magnetic pressure $H^2/8\pi$ be positive, it follows that the constants d and c must

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be such that the inequalities

$$d + nF'(0) > 0, \quad c + mF'(0) > 0$$

are satisfied.

The boundary, on which $p_* = $ const, will be an ellipse. If, at the initial time $t = 0$, we assign

$$f(0) = f_0 > 0, \quad \psi(0) = \psi_0 > 0, \quad f'(0) = \psi'(0) = 0. \tag{1.10}$$

then it will follow from the solution (1.8) that, at $t = 0$, we have $u = v = 0$. Both the hydrodynamic and the magnetic pressure at the initial time increase with the distance from the center of the ellipse. It follows from eqs.(1.7) that f'' and ψ'' will be negative, and since $f'(0) = \psi'(0) = 0$, then f' and ψ' for $t > 0$ will also be negative. For this reason, f and ψ will decline from the positive values f_0 and ψ_0 at $t = 0$ to zero values.

It is clear from the expression for the velocity components in the solution (1.8) and from our analysis of the behavior of the functions f and ψ that the motion under consideration will describe the compression of a plasma cylinder, at rest at the initial time, in the magnetic field.

In the general case, the system of differential equations (1.7) can be solved only by the numerical method. Simplifications are obtained by a special selection of the constants a, b and by an additional assumption with respect to the functions f and ψ.

If $a = b > 0$ and $f = \psi$, then the system of differential equations (1.7) reduces to a single equation which is integrated in quadratures:

$$f'' = -af^{-3}[nf^{2(2-\gamma)} + m]. \tag{1.11}$$

Let us prescribe the initial conditions in the form $f(0) = f_0$, $f'(0) = 0$. For simplicity, let $\gamma = 3/2$. Equation (1.11) is easily transformed into an equation of the first order, and for $\nu = 3/2$ we have

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$$(f')^2 = -\varkappa(f - f_0)(f - f_1)f^{-2}. \qquad (1.12)$$

where

$$\varkappa = \frac{a(m + 2nf_0)}{2f_0^2}, \qquad f_1 = \frac{-m}{m + 2nf_0}f_0.$$

Let us consider the case when $m < 0$, $n > 0$, $m + 2nf_0 > 0$. Then, $\varkappa > 0$ and $f_1 > 0$. The right-hand side of eq.(1.12) will therefore be positive if f has a value between f_0 and f_1.

Integrating eq.(1.12) under the condition that $f(0) = f_0$, we obtain

$$\sin\left[-\frac{2\sqrt{\varkappa}}{f_0 + f_1}\left(t \pm \frac{\sqrt{-(f - f_0)(f - f_1)}}{\sqrt{\varkappa}} \pm \frac{\varkappa}{2}\frac{f_0 + f_1}{2\sqrt{\varkappa}}\right)\right] = \frac{f_0 + f_1 - 2f}{\pm(f_0 - f_1)}. \qquad (1.13)$$

The plus sign is taken if $f_1 < f_0$; the minus sign, if $f_1 > f_0$.

The solution (1.13) shows that the function f is a periodic function of t, assuming values between f_0 and f_1.

In the presence of symmetry, the motion of any fluid particle of the gas will be described by the equation $r = f(t)r_0/f_0$, where r is the coordinate of the specified fluid particle at $t = 0$.

It follows from the behavior of the function f that the particle will oscillate between the two extreme positions r_0 and $r_1 = f_1 r_0/f_0$, i.e., there will be a pulsation of the plasma cylinder instead of a motion of the type of an accumulation of the gas at the origin of coordinates and a dispersion of the gas toward infinity.

We note that the oscillatory motions considered here are characteristic only of magnetic gas dynamics at $\nu \neq 2$. In fact, if in our solution (1.8) we set $m = 0$, $c = 0$, we will obtain the solution of the equations of conventional gas dynamics (Bibl.3). It will be clear from eqs.(1.7) that, at $m = c = 0$, /949 oscillatory motions without accumulation of gas at the origin of coordinates will be absent.

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At $\gamma = 2$, eqs.(1.5) permit a separation of variables without bounding at h_0, which we did in eq.(1.6). In this case, the solution of the system of equations (1.1) will be found to depend on two arbitrary functions, and to have the form

$$u = \frac{f'}{f} x, \quad v = \frac{\psi'}{\psi} y, \quad p = f^{-2}\psi^{-2}\left[d + nF(\alpha) - \frac{h_0^2\left(\frac{x}{f}, \frac{y}{\psi}\right)}{8\pi}\right],$$

$$H = f^{-1}\psi^{-1}h_0\left(\frac{x}{f}, \frac{y}{\psi}\right), \quad \rho = f^{-1}\psi^{-1}F'(\alpha), \quad \alpha = \frac{a}{2}\frac{x^2}{f^2} + \frac{b}{2}\frac{y^2}{\psi^2}.$$

where $h_0(x/f, y/\psi)$ and $F(\alpha)$ are arbitrary functions of their arguments, while n, d, a, and b are arbitrary constants.

The functions f and ψ satisfy the system of equations

$$f'' + anf^{-2}\psi^{-1} = 0, \quad \psi'' + bn\psi^{-2}f^{-1} = 0.$$

2. Up to now we have assumed the magnetic field to be directed along the Oz axis and the gas to be without motion in that direction.

Let us now consider the case when uniform expansion of the gas takes place along the magnetic field, i.e., when the velocity component along the Oz axis is of the form $w = z/(t + t_0)$. The quantities p, ρ, and H do not depend on the coordinate z. Under these assumptions, the system of equations of magnetic gas dynamics, describing motion in the plane xOy, will be the system of equations (1.1) at $\nu = 1$.

Proceeding in analogy to Section 1, we may obtain the following solution of the system of equations (1.1):

$$u = \frac{f'}{f} x, \quad v = \frac{\psi'}{\psi} y, \quad \rho = f^{-1}\psi^{-1}(t + t_0)^{-1}F'(\alpha),$$

$$p = f^{-\gamma}\psi^{-\gamma}(t + t_0)^{-\gamma}[d + nF(\alpha)], \quad H^2 = 8\pi f^{-2}\psi^{-2}[c + mF(\alpha)], \quad (2.1)$$

$$\alpha = \frac{a}{2}\frac{x^2}{f^2} + \frac{b}{2}\frac{y^2}{\psi^2}$$

where $F(\alpha)$ is an arbitrary function, while a, b, c, n, m, d, t_0 are arbitrary constants. The functions f and ψ are the solution of the system of equations

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$$f'' + axf^{-\gamma}\psi^{1-\gamma}(t+t_0)^{1-\gamma} + axf^{-2}\psi^{-1}(t+t_0) = 0,$$
$$\psi'' + bx\psi^{-\gamma}f^{1-\gamma}(t+t_0)^{1-\gamma} + bx\psi^{-2}f^{-1}(t+t_0) = 0.$$

It can be proved by direct substitution that eq.(2.1) is a solution of the system of equations (1.1).

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